ON FINITE DIMENSIONAL SUBSPACES OF BANACH SPACES

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ABSTRACT

The common Banach spaces are investigated with respect to some properties of their finite dimensional subspaces.

1. It is well-known (by the Hahn-Banach theorem) that for each element x in a Banach space X one can find a functional $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$. The following natural question arises: Given a finite dimensional subspace $E \subset X$, is it possible to find a finite dimensional subspace $F \subset X^*$ such that for each $x \in E$ $\|x\| = \sup_{f \in S_F} |f(x)|$?

In this paper we show that the answer is negative, and investigate some similar properties concerning finite dimensional subspaces. We prove that the spaces $c_0(S)$ and real L_p, where p is an even integer, satisfy the above-mentioned condition, while C and L_p for all other p's do not satisfy it.

The terminology and notations are generally the same as in [1]. S_X denotes the closed unit ball of the Banach space X. If F is a subspace of a conjugate space X^* then $F_\perp = \{x : x \in X, f(x) = 0 \text{ for all } f \in F\}$. If $E \subset X$ then $E^\perp = \{f : f \in X^*, f(x) = 0 \text{ for all } x \in E\}$.

2. Let us discuss the following conditions on an infinite dimensional normed space X, concerning its finite dimensional subspaces:

(1) For every finite dimensional subspace $E \subset X$ there exists a subspace $F \subset X^*$ such that for each $x \in E$ $\|x\| = \sup_{f \in S_F} |f(x)|$ and $F_\perp$ is infinite dimensional.

(2) For every finite dimensional subspace $E \subset X$ it is possible to find a finite dimensional $F \subset X^*$ such that for each $x \in E$ $\|x\| = \sup_{f \in S_F} |f(x)|$.

(3) For every finite dimensional subspace $E \subset X$ there exist a finite dimensional subspace $G \subset X$ such that $E \subset G$ and a projection P_G of X onto G with $\|P_G\| = 1$.

Denote by $\mathscr{A}_i$ the class of all normed spaces satisfying condition (i) $i = 1,2,3$. It is obvious that if we require in (1) that F is w^*-closed $\mathscr{A}_1$ will not change. Of course, $\mathscr{A}_1 \supseteq \mathscr{A}_2 \supseteq \mathscr{A}_3$. In §4, 7 we shall show that $\mathscr{A}_1 \supsetneqq \mathscr{A}_2 \supsetneqq \mathscr{A}_3$.

The following two lemmas give equivalent conditions to (1) and (2) respectively.

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LEMMA 1. *Let X be a normed space; then $X \in \mathscr{A}_1$ if and only if for every finite dimensional subspace $E \subset X$ there exists an infinite dimensional closed subspace $G \subset X$ satisfying the following conditions*:

(a) $G \cap E = \{0\}$.

(b) *If H is the subspace spanned by the elements of E and G, then there exists a projection P_E of H onto E along G.*

(c) $\| P_E \| = 1$.

Proof. Necessity: Take for G the subspace $F_{\perp}$. If $x \in G \cap E$ then $\| x \| = \sup_{f \in S_F} | f(x) | = 0$, hence $x = 0$; if $x = e + y$, $e \in E$, $y \in G$ then $\| e + y \| \geqq \sup_{f \in S_F} | f(e + y) | = \sup_{f \in S_F} | f(e) | = \| e \|$. Hence, the transformation P_E defined by $P_E(e + y) = e$ is a projection of H onto E along G and since $\| e \| \leqq \| e + y \|$, $\| P_E \| = 1$. Sufficiency: For every $f \in E^*$ define $\bar{f}(e + y) = f(e)$ for each $e + y \in H$. Now, $\bar{f} \in H^*$, and $\bar{f}(y) = 0$ for every $y \in G$. By (b) and (c) $\|\bar{f}\|_H = \sup_{\|e+y\|=1} |\bar{f}(e + y)| = \sup_{\|e\| \leqq 1} |\bar{f}(e)| = \| f \|_E$. It follows that for every $x \in E$ $\|x\| = \sup_{\bar{f} \in S_{H^*}} |\bar{f}(x)|$. Denote by F the closed subspace of X^* spanned by all Hahn-Banach extensions of the $\bar{f}$'s to the whole space X. It is obvious that for each $x \in E$ $\|x\| = \sup_{f \in S_F} |f(x)|$. Since $G \subset F_{\perp}$, $F_{\perp}$ is infinite dimensional.

LEMMA 2. *Let X be a normed space. Then $X \in \mathscr{A}_2$ if and only if for every finite dimensional subspace E there exists a closed subspace $G \subset X$ satisfying the conditions* (a) (b) *and* (c) *of Lemma* 1 *and condition*,

(d) *The co-dimension of G in X is finite.*

Proof. The necessity is clear by the proof of Lemma 1. Sufficiency: By [1] p. 25, Lemma 1, if T is the natural mapping of X onto X/G, defined by $Tx = x + G$, then T^* is a linear isometry of $(X/G)^*$ onto $G^{\perp} \cap X^* = F$. Since, by (d), X/G is finite dimensional, so is F. By (b) and (c), if $x \in E$ $\|x + G\| = \inf_{y \in G} \| x + y \| \geqq \|x\|$. On the other hand $\| x + G \| \leqq \| x \|$, hence $\| x \| = \| x + G \|$. For every $x \in E$ there exists a functional $g \in (X/G)^*$ such that $\| g \| = 1$ and $g(x + G) = \| x + G \|$. But $T^*g \in F$, and

$$(T^*g)(x) = g(Tx) = g(x + G) = \| x + G \| = \| x \|.$$

T^* is isometric so $\| T^*g \| = \| g \| = 1$. It follows that for each $x \in E$

$$\| x \| = \sup_{f \in S_F} | f(x) |.$$

REMARK 1. The class $\mathscr{A}_3$ is not empty; every Hilbert space space belongs to $\mathscr{A}_3$.

REMARK 2. According to [3] if $X \in \mathscr{A}_2$, and if for every n dimensional subspace $E \subset X$ the dimension of F is also n then X is a Hilbert space.

REMARK 3. If $X \in \mathscr{A}_2$ then every subspace $Y \subset X$ also belongs to $\mathscr{A}_2$. But as shown in §4 there exists a Banach space X and a closed subspace $Y \subset X$ such that $X \in \mathscr{A}_1$ while $Y \notin \mathscr{A}_1$.

3. We have mentioned that $l_2 \in \mathscr{A}$. Let us discuss now other examples of members of $\mathscr{A}_3$.

A Banach space X is called polyhedral (see [4]) if every finite dimensional subspace of X has a polyhedron as its unit ball. J. Lindenstrauss proved in [6, p. 100, Corollary 2] that every polyhedral Banach space X for which X^{**} is a P_1 space belongs to $\mathscr{A}_3$. In fact he proved that every finite dimensional subspace of X is contained in a finite dimensional subspace which is a P_1 space, that is, isometric to l^n_∞ for a suitable n.

The following converse implication is also a consequence of the theory of J. Lindenstrauss [6]:

LEMMA 3. *If $X \in \mathscr{A}_3$ and X^{**} is a P_1-space then X is polyhedral.*

Proof. Let $E \subset X$ be of finite dimension. There exists a projection P of X onto a finite dimensional subspace G which contains E, such that $\| P \| = 1$.

From [6] p. 16 Corrollary 3 it follows that G is a P_1 space, hence, it is isometric to a space l^n_∞, so the unit ball of E is a polyhedron.

In connection with polyhedral spaces, let us prove

LEMMA 4. *If a Banach space X is polyhedral than $X \in \mathscr{A}_2$.*

Proof. Let $E \subset X$ be a finite dimensional subspace. Denote by $f_1, f_2, \cdots, f_k$ the extreme points of S_{E^*}, and by $\bar{f}_j$ any Hahn-Banach extension of f_j to the whole space X; $1 \leqq j \leqq k$. Obviously, any element of E attains its norm on the unit ball of the subspace spanned by $\bar{f}_1, \bar{f}_2, \cdots, \bar{f}_k$.

In [4] V. Klee proves that c_0 is polyhedral. Since m is a P_1 space, it follows from the preceding remarks that $c_0 \in \mathscr{A}_3$. We shall give here a direct proof of a slightly stronger result for c_0.

Let $\{e_k\}_{k=1}^\infty$ be the unit vectors basis in c_0. Let us denote by E_k the closed subspace spanned by the first k unit vectors: $E_k = [e_1, e_2, \cdots, e_k]$. E^k will denote the closed subspace $[e_{k+1}, e_{k+2}, \cdots]$. Denote by P_k the projection of c_0 onto E_k, defined by $P_k(\sum_{i=1}^\infty \gamma_i e_i) = \sum_{i=1}^k \gamma_i e_i$. It is known that $\| P_k \| = \| I - P_k \| = 1$ for all k.

THEOREM 1. *Let $\varepsilon < 1/2$ be a positive number, and E a finite dimensional subspace of c_0. Then there exists a finite dimensional closed subspace G of c_0 with the following properties:*

(a) $E \subset G \subset c_0$.

(b) *If $\dim(G) = n$ then there exists a linear isometry T of E_n onto G.*

(c) *The transformation $P_G = TP_n$ is a projection of c_0 onto G and $\| P_G \| = 1$.*

(d) $\| I - P_G \| \leqq 1 + \varepsilon$.

(e) $(I - P_G)(c_0) = E^n$.

Proof. If E is k-dimensional, then by [7, Theorem 2] we can find a basis $x_1, x_2, \cdots, x_k$ in E and functionals $f_1, f_2, \cdots, f_k$ in X^* such that $f_i(x_j) = \delta_{ij}$ and $\|f_i\| = \|x_i\| = 1$ for $1 \leqq i \leqq k$. If $x_j = \sum_{i=1}^{\infty} \alpha_i^j e_i$ is the representation of x_j with respect to the usual basis $\{e_i\}_{i=1}^{\infty}$ then an integer n can be found, such that the following conditions are satisfied:

I. $|\alpha_i^j| \leqq \varepsilon \cdot k^{-3}$ for all $i > n$ and $1 \leqq j \leqq k$.

II. The elements $x_j' = P_n x_j = \sum_{i=1}^{n} \alpha_i^j e_i$ $1 \leqq j \leqq k$ are linearly independent.

Let us denote by E' the subspace spanned by $x_1', x_2', \cdots, x_k'$. Let U be the transformation of E to E' defined by $U(\sum_{i=1}^{k} \beta_i x_i) = \sum_{i=1}^{k} \beta_i x_i'$. U is a linear one-to-one transformation from E onto E'. Let us show that U is isometric.

If $\| \sum_{j=1}^{k} \beta_j x_j \| = 1$ then

$$1 = \left\| \sum_{j=1}^{k} \beta_j x_j \right\| = \left\| \sum_{j=1}^{k} \beta_j \left(\sum_{i=1}^{\infty} \alpha_i^j e_i \right) \right\| = \left\| \sum_{i=1}^{\infty} \left(\sum_{j=1}^{k} \beta_j \alpha_i^j \right) e_i \right\|$$

$$= \sup_{1 \leqq i < \infty} \left\{ \left| \sum_{j=1}^{k} \beta_j \alpha_i^j \right| \right\} = \sup_{1 \leqq i \leqq n} \left\{ \left| \sum_{j=1}^{k} \beta_j \alpha_i^j \right| \right\}.$$

The last equality follows from the following considerations: Since $\|f_j\| = 1$ and $\| \sum_{i=1}^{k} \beta_i x_i \| = 1$ it follows that $|\beta_j| = |f_j(\sum_{i=1}^{k} \beta_i x_i)| \leqq 1$. Hence, by I, for $i > n$ $|\sum_{j=1}^{k} \beta_j \alpha_i^j| \leqq (\sup_{1 \leqq j \leqq k} |\beta_j|) \cdot (\sum_{j=1}^{k} |\alpha_i^j|) \leqq k \cdot \varepsilon \cdot k^{-3} = \varepsilon \cdot k^{-2} \leqq 1/2$. But $\sup_{1 \leqq i < \infty} \{ |\sum_{j=1}^{k} \beta_j \alpha_i^j| \} = 1$, therefore the sup is attained from $1 \leqq i \leqq n$. We have just proved that

$$1 = \sup_{1 \leqq i \leqq n} \left\{ \left| \sum_{j=1}^{k} \beta_j \alpha_i^j \right| \right\} = \left\| \sum_{i=1}^{n} \left(\sum_{j=1}^{k} \beta_j \alpha_i^j \right) e_i \right\|$$

$$= \left\| \sum_{j=1}^{k} \beta_j \left(\sum_{i=1}^{n} \alpha_i^j e_i \right) \right\| = \left\| \sum_{j=1}^{k} \beta_j x_j' \right\| = \left\| U \left(\sum_{j=1}^{k} \beta_j x_j \right) \right\|.$$

It follows that U is a linear isometry from E onto E'. E' is a k-dimensional subspace of E_n, so there exists a projection Q of E_n onto E' with $\|Q\| \leqq k$. Let T be the transformation from E_n into c_0, defined by $T = U^{-1}Q + I - Q$. T is linear. We are going to prove that $\| \sum_{i=1}^{n} \gamma_i e_i \| = 1$ if and only if $\| T(\sum_{i=1} \gamma_i e_i) \| = 1$. Suppose that $\| \sum_{i=1}^{n} \gamma_i e_i \| = 1$, $Q(\sum_{i=1}^{n} \gamma_i e_i) = \sum_{j=1}^{k} \beta_j x_j'$, and that $(I - Q)(\sum_{i=1}^{n} \gamma_i e_i) = \sum_{i=1}^{n} \delta_i e_i$. It follows that $T(\sum_{i=1}^{n} \gamma_i e_i) = \sum_{j=1}^{k} \beta_j x_j + \sum_{i=1}^{n} \delta_i e_i$. Now, $\|U\| = \|U^{-1}\| = 1$ and $\|Q\| \leqq k$ so $|\beta_j| = |f_j[U^{-1}Q(\sum_{i=1}^{n} \gamma_i e_i)]| \leqq k$, hence, by I we get for i $> n$.

III. $|\sum_{j=1}^{k} \beta_j \alpha_i^j| \leqq k(\sum_{j=1}^{k} |\alpha_n^j|) \leqq k^2 \varepsilon \cdot k^{-3} < 1/2$ therefore $1 = \| \sum_{i=1}^{n} \gamma_i e_i \| = \| \sum_{j=1}^{k} \beta_j x_j' + \sum_{i=1}^{n} \delta_i e_i \| = \| \sum_{i=1}^{n} (\sum_{j=1}^{k} \beta_j \alpha_i^j + \delta_i) e_i \| = \sup_{1 \leqq i \leqq n} \{ |\sum_{j=1}^{k} \beta_j \alpha_i^j + \delta_i| \} = \sup_{1 \leqq i < \infty} \{ |\sum_{j=1}^{k} \beta_j \alpha_i^j + \bar{\delta}_i| \}$, where

$$\bar{\delta}_i = \begin{cases} \delta_i \text{ for } i \leqq n \\ 0 \text{ for } i > n \end{cases}$$

(The last equality follows from II). But

$$\sup_{1\leq i<\infty}\left\{\left|\sum_{j=1}^{k}\beta_j\alpha_i^j+\delta_i\right|\right\}=\left\|\sum_{j=1}^{k}\beta_jx_j+\sum_{i=1}^{n}\delta_ie_i\right\|=\left\|T\left(\sum_{i=1}^{n}\gamma_ie_i\right)\right\|,$$

so we have proved that if $\| \sum_{i=1}^{n}\gamma_ie_i\|=1$ then $\|T(\sum_{i=1}^{n}\gamma_ie_i)\|=1$. A similar method yields the converse implication. An immediate conclusion is the following: $G=T(E_n)$ is n-dimensional and T is a linear isometry from E_n onto G. The definition of T ensures that $E=T(E')\subset G$.

If, again, $\sum_{i=1}^{n}\gamma_ie_i=\sum_{j=1}^{k}\beta_jx_j'+\sum_{i=1}^{n}\delta_ie_i$ then $P_nTP_n(\sum_{i=1}^{\infty}\gamma_ie_i)=P_nT(\sum_{i=1}^{n}\gamma_ie_i)=P_n(\sum_{j=1}^{k}\beta_jx_j+\sum_{i=1}^{n}\delta_ie_i)=\sum_{j=1}^{k}\beta_jx_j'+\sum_{i=1}^{n}\delta_ie_i=\sum_{i=1}^{n}\gamma_ie_i=P_n(\sum_{i=1}^{\infty}\gamma_ie_i)$, hence $P_nTP_n=P_n$, and so $TP_nTP_n=TP_n$. It follows that $P_G=TP_n$ is a projection of c_0 onto $G=T(E_n)$, with $\|P_G\|=1$; (for $1\leqq\|TP_n\|\leqq\|T\|\cdot\|P_n\|=1$). Let us remark that the definition of T and the same methods yield the following inequality: For each $\sum_{i=1}^{n}\gamma_ie_i\in E_n$ $\|\sum_{i=1}^{n}\gamma_ie_i-T(\sum_{i=1}^{n}\gamma_ie_i)\|\leqq\varepsilon\|\sum_{i=1}^{n}\gamma_ie_i\|$. It follows that $\|I-TP_n\|=\|I-P_n+P_n-TP_n\|\leqq\|I-P_n\|+\|P_n\|\cdot\|I_0-T\|\leqq1+\varepsilon$, where I_0 denotes the identity on E_n. The proof of (e) is left to the reader.

Corollary 1. *Theorem* 1, *with slight modifications in its formulation, remains true if we replace* c_0 *by* $c_0(S)$, *where* S *is an infinite set.*

In fact, if $E\subset c_0(S)$ is of finite dimension, then there exists a sequence $S_0\subset S$ such that $x(s)=0$ for every $x\in E$ and $s\in S-S_0$. Denote by the subspace

$$\{x:x\in c_0(S),\ x(s)=0\ \text{for all}\ s\in S-S_0\}.$$

The natural projection P of $c_0(S)$ onto H is of norm 1, and so is $I-P$. H is isometric to c_0, so by Theorem 1 there exists a projection P_G of H onto a finite dimensional closed subspace G which contains E, with $\|P_G\|=1$. P_GP is a projection of norm 1 of $c_0(S)$ onto G. By the same Theorem 1, G is isometric to a space l_∞^n. It is easy to see that since $\|I-P_G\|\leqq1+\varepsilon$ (I_0 is the identity on H) $\|I-P_GP\|=\|(I_0-P_G)P+I-P\|=\max\{\|(I_0-P_G)P\|,\ \|I-P\|\}\leqq\max\{1+\varepsilon,1\}=1+\varepsilon$.

Remark 4. The properties (a) to (e) do not characterize c_0. It is easy to prove Theorem 1 for the space $(R_1\oplus R_2\oplus R_3\cdots)_{c_0}=X$ instead of c_0. (R_k denotes the k-dimensional euclidean space.) J. Lindenstrauss proved in [5] that X is not isomorphic to c_0.

4. Now we shall show that the space c has not the property (1). From this will follow that no infinite dimensional space $C(S)$ with S compact Hausdorff has the property (2). This proves also that, as was expected, none of the properties discussed here is invariant under isomorphisms.

Let $a=\{a_i\}_{i=1}^{\infty}$ be an element of c such that $a_i \neq a_j$ if $i \neq j$ and $0 \leqq a_i \leqq 1$, $a_0 = \lim_{i\to\infty} a_i \neq a_j$ for $j=1,2,3,\cdots$. We shall consider the plane determined by the null vector, the given a and the vector $\{(1-a_i^2)^{1/2}\}_{i=1}^{\infty}$. The family of vectors $[\{a_n a_i + (1-a_n^2)^{1/2}(1-a_i^2)^{1/2}\}_{i=1}^{\infty}]$ $n=0,1,2,\cdots$ is contained in the plane and the only point of the unit ball of l_1 at which the nth vector of the family attains its norm is the $n+1$th unit vector of l_1. This follows from the fact that the nth coordinate (the limit) of $\{a_n a_i + (1-a_n^2)^{1/2}(1-a_i^2)^{1/2}\}_{i=1}^{\infty}$ is 1 where $n \geqq 1 (n=0)$ and the other are positive but less than 1. Hence, the smallest closed subspace of l_1 on the unit ball of which every vector from our plane attains its norm is l_1 itself.

The same plane taken this time in the space m shows that this space also has not the property (1). (We use the fact that l is w^*-dense in m^*).

Considering in $C[0,1]$ the subspace spanned by $f_1(x)=x$, $f_2(x)=1-x^2$ it can be proved that $C[0,1]$ does not satisfy condition (1).

Now we are able to display a space which belongs to $\mathscr{A}_1$ but fails to belong to $\mathscr{A}_2$. If X is a space which does not belong to $\mathscr{A}_2$, the space $Y=(l_2 \oplus X)_{l_1}$ is the desired one. Indeed, the conjugate of Y is $(l_2 \oplus X^*)_m$ and if E is a finite dimensional subspace of Y any element of E attains its norm on the unit ball of $(P(E)\oplus X^*)_m \subset (l_2 \oplus X^*)_m$, where P denotes the natural projection of Y onto l_2. The annihilator of $P(E) \oplus X^*$ is the orthogonal complement of $P(E)$ in l_2 which is obviously infinite dimensional. But Y fails to have the property (2) since its subspace X has not this property. (See Remark 3). If we suppose that X does not belong even to $\mathscr{A}_1$, like the space c for instance, we see from the above example that property (1) of a space is not inherited by its subspaces.

5. We shall prove now that the property (1) is not valid in l_1. Thus no infinite dimensional L-space has property (2). Let $a=\{a_i\}_{i=1}^{\infty}$, $b=\{b_i\}_{i=1}^{\infty}$ be two elements of l_1 such that

$$\frac{a_i}{b_i} > \frac{a_{i+1}}{b_{i+1}} \quad , \quad b_i > 0$$

for every i and consider the subspace generated by them. For a given natural number n let α_n be a scalar such that $(a_n/b_n) > \alpha_n > (a_n+1)/(b_n+1)$. The vector $a-\alpha_n b$ attains its norm on the unit ball of m at the point $\{\operatorname{sign}(a_i-\alpha_n b_i)\}_{i=1}$ and we have:

$$\operatorname{sign}(a_i-\alpha_n b_i)=1 \qquad 1 \leqq i \leqq n$$

$$\operatorname{sign}(a_i-\alpha_n b_i)=-1 \qquad i \geqq n+1.$$

Choosing a scalar α_0 such that $\alpha_0 > (a_1/b_1)$ the vector $\{a_i-\alpha_0 b_i\}_{i=1}^{\infty}$ will attain its norm only at the point $(-1,-1,-1,\cdots)$ of S_m. The smallest w^*-closed subspace of m which contains all the points of m found above is m itself, and this proves our statement.

The space $L_1[0,1]$ fails also to have property (1). To see this it is enough to take the subspace generated by $f_1(x)=1$ and $f_2(x)=x$ and to use the fact that the characteristic functions of intervals form a total family over $L_1[0,1]$.

One may ask if the completion of a normed space satisfying one of the conditions (1), (2), (3) must satisfy this condition. The answer is negative since the linear subspace of l_1 generated by the unit vectors is a member of $\mathscr{A}_3$ while its completion fails to belong even to $\mathscr{A}_1$.

THEOREM 2. *The spaces* l_p $1 \leqq p < \infty$, *where* p *is not an even integer, do not belong to the class* $\mathscr{A}_2$.

THEOREM 3. *If* (S,Σ,μ) *is a measure space, the real space* $L_p(S,\Sigma,\mu) = L_p$ *with* p *an even integer belongs to* $\mathscr{A}_2$.

In proving these theorems we shall constantly make use of the known facts that the function $f(t)$ of $L_p(p > 1)$ atains its norm on the unit ball of $L_q(1/p+1/q=1)$ at the point $\|f\|^{-p} \cdot |f(t)|^{p-1}$ sign $f(t)$ and only at this point. We shall refer everywhere to $|f(t)|^{p-1}\operatorname{sign} f(t)$ since this function belongs to the same subspaces of L_q as $\|f\|^{1-p} \cdot |f(t)|^{p-1}\operatorname{sign} f(t)$ does.

Proof of Theorem 2. Let n be any integer greater than p. (If p is not an integer n may be any integer greater than 1). We shall consider the two dimensional subspace E_n of l_p^n spanned by $a_n = (1,1,\cdots 1)$ and $b_n = (1,2,3,\cdots,n)$ and we shall show that there is no proper subspace of l_q^n which contains all the points of the unit ball having a support hyperplane generated by an element of E_n.

Assume that n is an odd integer. Denote

$$P_i(\lambda) = (1+\lambda i)^{p-1} = |1+\lambda i|^{p-1} \qquad 1 \leqq i \leqq n.$$

Since $a_n + \lambda b_n \in E_n$ for any real λ, in order to prove the assertion it will be enough to display n independent vectors of the the form

$$\{P_1(\lambda)\operatorname{sign}(1+\lambda),\ P_2(\lambda)\operatorname{sign}(1+2\lambda),\cdots,P_n(\lambda)\operatorname{sign}(1+n\lambda)\}.$$

We shall give to λ n different values submitted to the conditions:

$$\lambda_i \geqq 0 \qquad 1 \leqq i \leqq p$$

$$-\frac{1}{n-j} < \lambda_{p+j} < -\frac{1}{n-j+1} \qquad 1 \leqq j \leqq n-p.$$

The corresponding vectors are the rows of the matrix:

$$\begin{vmatrix} P_1(\lambda_1), & P_2(\lambda_1), \cdots & & , P_n(\lambda_1), \\ P_1(\lambda_2), & P_2(\lambda_2), \cdots & & , P_n(\lambda_2), \\ \vdots & \vdots & & \vdots \\ P_1(\lambda_p), & P_2(\lambda_p), \cdots & & , P_n(\lambda_p), \\ P_1(\lambda_{p+1}), & P_2(\lambda_{p+1}), \cdots & , P_{n-1}(\lambda_{p+1})), & -P_n(\lambda_{p+1}), \\ P_1(\lambda_{p+2}), & P_2(\lambda_{p+2}), \cdots, P_{n-2}(\lambda_{p+2}), & -P_{n-1}(\lambda_{p+2}), & -P_n(\lambda_{p+2}), \\ \vdots & \vdots & & \vdots \\ P_1(\lambda_n), & P_2(\lambda_n), \cdots, P_p(\lambda_n), & -P_{p+1}(\lambda_n) - \cdots & -P_n(\lambda_n), \end{vmatrix}$$

If its rank is less than n, there exist n scalars μ_k $1 \leqq k \leqq n$ such that

$$\text{I} \qquad \sum_{k=1}^{n} \mu_k P_k(\lambda_i) = 0 \qquad 1 \leqq i \leqq p$$

$$\text{II} \qquad \sum_{k=1}^{n-j} \mu_k P_k(\lambda_{p+j}) - \sum_{l=n-j}^{n} \mu\mu_l P_l(\lambda_{p+j}) = 0 \qquad 1 \leqq j \leqq n-p$$

Since the degree of $P_k(\lambda)$ is $p-1$, we deduce from I that

$$\text{III} \qquad \sum_{k=1}^{n} \mu_k P_k(\lambda) = 0$$

for any λ. Taking in II $j = 1$, in III $\lambda = \lambda_{p+1}$ and subtracting one equality from the other we get $\mu_{p+1} = 0$. Continuing in this way we get $\mu_{p+j} = 0$ for $1 \leqq j \leqq n-p$. Hence, III may be written as

$$\text{IV} \qquad \sum_{k=1}^{p} \mu_k P_k(\lambda) = 0.$$

But it is easy to see that these polynomials are linearly independent, and this proves our assertion about l_p^n where p is an odd integer.

Now suppose that p is not an integer. The proof of our assertion in this case will be achieved if we show that n positive numbers $\lambda_1, \lambda_2, \cdots \lambda_n$ can be found such that $D_n(\lambda_1, \lambda_2, \cdots, \lambda_n) = \det(1 + i\lambda_j)^{p-1} \neq 0$. For $n = 2$ it is easy to check that if $\lambda_1 \neq \lambda_2$ then $D_2(\lambda_1, \lambda_2) \neq 0$. Suppose that $n-1$ positive numbers $\lambda_1, \lambda_2, \cdots, \lambda_{n-1}$ were found such that

$$\text{V} \qquad D_{n-1}(\lambda_1, \lambda_2, \cdots, \lambda_{n-1}) \neq 0$$

but

$$\text{VI} \qquad D_n(\lambda_1, \lambda_2, \cdots, \lambda_{n-1}, \lambda) \neq 0$$

for any $\lambda > 0$. Differentiating VI $n-1$ times we get

$$\begin{vmatrix} (1+\lambda_1)^{p-1}, & (1+2\lambda_1)^{p-1}, \cdots & (1+n\lambda_1)^{p-1} \\ \vdots & & \vdots \\ (1+\lambda_{n-1})^{p-1}, & (1+2\lambda_{n-1})^{p-1}, \cdots & (1+n\lambda_{n-1})^{p-1} \\ (1+\lambda)^{p-1-j}, & 2^j(1+2\lambda)^{p-1-j}, \cdots & n^j(1+n\lambda)^{p-1-j} \end{vmatrix} = 0$$

$$j = 1,2,\cdots,n-1.$$

These equalities together with VI form a linear system of equations the coefficients being the elements of the last rows of the determinants. From V we deduce that this system has non-trivial solutions. Therefore

$$\begin{vmatrix} (1+\lambda)^{p-1}, & (1+2\lambda)^{p-1}, \cdots & ,(1+n\lambda)^{p-1} \\ (1+\lambda)^{p-2}, & 2(1+2\lambda)^{p-2}, \cdots & ,n(1+n\lambda)^{p-2} \\ \vdots & \vdots & \vdots \\ (1+\lambda)^{p-n}, & 2^{n-1}(1+2)^{p-n}, \cdots & n^{n-1}(1+n\lambda)^{p-n} \end{vmatrix} = 0$$

for any $\lambda > 0$. This equality is obviously false and, consequently VI cannot be true for any $\lambda > 0$.

Now we shall construct a two-dimensional subspace of l_p with the property that its elements attain their norms on the unit ball of l_q at points which span an infinite dimensional subspace of l_q. This plane is spanned by the vectors a and b constructed as follows: The nth block of n coordinates in a is

$$n^{-2} \cdot (1 + 2^p + 3^p + \cdots + n^p)^{-1/p} \cdot (1,1,1,\cdots,1),$$

and in b is $n^{-2} \cdot (1 + 2^p + 3^p + \cdots n^p)^{-1/p} \cdot (1,2,3,\cdots,n)$, for $n = 1,2,3,\cdots$. From the above discussion we know that this plane has the required property.

Proof of Theorem 3. We shall prove that for any n-dimensional subspace F of L_p there exists a subspace F of L_q of dimension C^{n-1}_{n+p-2} such that $\|x\| = \sup_{f \in S_F} |f(x)|$ for any $x \in E$. Let $x_1(t)$, $x_2(t),\cdots,x_n(t)$ be a basis of E. We have to detemine the dimension of the space spanned by the functions $(\sum_{i=1}^n \lambda_i x_i(t))^{p-1} = |\sum_{i=1}^n \lambda_i x_i(t)|^{p-1} \cdot \operatorname{sign}(\sum_{i=1}^n \lambda_i x_i(t))$. These functions are linear combinations of the functions $x_1^{k_1} \cdot x_2^{k_2} \cdots x_n^{k_n}$ for all possible non-negative integers $k_1, k_2, \cdots, k_{3n}$ which satisfy $\sum_{i=1}^n k_i = p-1$. Any function of this type belongs to L_q (this can be seen if we use the fact that L_q is a lattice) and there are C^{n-1}_{p+n-2} such functions. Consequently all the functions $(\sum_{i=1}^n {}_i x_i(t))^{p-1}$ lie in a subspace of L_q which has the dimension C^{n-1}_{p+n-2}.

An immediate consequence of Theorem 2 is the fact that no space L_p (μ) with $p \neq 2k$, $k = 1,2,3,\cdots$ satisfies condition (2).

7. Let us discuss now reflexive Banach spaces X for which S_{X^*} is strictly convex. By [8] and [2], the last property is equivalent to the following one: Every bounded linear functional f defined on a subspace E of X has one and only one Hahn-Banach extension $\bar{f}$.

LEMMA 5. *If a reflexive space X belongs to $\mathscr{A}_3$, and S_{X^*} is strictly convex then $X^* \in \mathscr{A}_3$.*

Proof. Again by [7], given a finite dimensional subspace $F \subset X^*$, one can find a basis $f_1, f_2, \cdots, f_k$ for F and functionals $x_1, x_2, \cdots, x_k$ in X such that $\| x_i \| = \| f_i \| = 1$ and $f_i(x_j) = \delta_{ij}, 1 \leqq i \leqq k \; 1 \leqq j \leqq k$. Denote by E the subspace of X spanned by $x_1, x_2, \cdots, x_k$. Since $X \in \mathscr{A}_3$, there exists a subspace $G \subset X$ satisfying the following conditions:

(a) G is of finite dimension,

(b) $E \subset G$,

(c) There exists a projection P_G of X onto G.

(d) $\| P_G \| = 1$.

Denote $(I - P_G)(X) = H$. Each $x \in X$ has a unique representation $x = y + z$ where $y \in G$ and $z \in H$. For every element $\phi \in G^*$ define: $f_\phi(y + z) = \phi(y)$. f_ϕ is an extension of ϕ, it is linear and by (d) $\| f_\phi \| = \sup_{\|y+z\| \leqq 1} |f_\phi(y + z)| = \sup_{\|y\| \leqq 1} |\phi(y)| = \| \phi \|$.

Denote by M the subspace $\{f_\phi : \phi \in G^*\}$ of X^*. It is easy to see that M is isometric to G^* and so it is of finite dimension. Moreover, there exists a projection Q of X^* onto M, with $\| Q \| = 1$. For $1 \leqq i \leqq k \; f_i \in M$, because otherwise the restriction ϕ_i of f_i to E would admit two Hahn-Banach extensions $-f_i$ and f_{ϕ_i}. But this is impossible by the hypothesis of the lemma. It follows that $F \subset M$.

COROLLARY. *For $p = 2n$ (n an integer $\geqq 2$) $L_p \in \mathscr{A}_2$ but $L_p \notin \mathscr{A}_3$.*

Proof. If $L_p \in \mathscr{A}_3$ then $L_q \in \mathscr{A}_3$ where $1/p + 1/q = 1$. But by Theorem 2 $L_q \notin \mathscr{A}_3$, since $q = p/p - 1 = 2n/(2n - 1)$ is not an integer.

REFERENCES

1. M. M. Day, *Normed linear spaces*, Springer Verlag, 1958.
2. S. R. Foguel, *On a theorem of A. E. Taylor*, Proc. Amer. Math. Soc. **9** (1958), 325.
3. S. Kakutani, *Some characterizations of euclidean space*, Jap. J. Math. **16** (1939), 93.
4. V. Klee, *Polyhedral sections of convex bodies*, Acta Math. **103** (1960), 243.
5. J. Lindenstrauss, *On some subspaces of c_0 and l_1*, Bull. Res. Council of Israel, **10F** (1961), 74.
6. ———, *Extension of compact operators*, Memoirs Amer. Math. Soc. **48** (1964).
7. A. E. Taylor, *A geometric theorem and its application to biorthogonal systems*, Bull. Amer. Math. Soc. **53** (1947), 614.
8. ———, *The extension of linear functionals*, Duke Math. J. **5** (1939), 538.